

Unified First Law and Some Comments

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Abstract

In this small article, unified first law has been analyzed and some results have been deduced from it. The results have been presented in the form of lemmas and some conclusions have been drawn from them.

The joint venture of quantum mechanics and general relativity leads to a remarkable discovery that a "black hole (BH) behaves like a black body". The thermal radiation emitted by a BH has temperature proportional to its surface gravity at the horizon while entropy is proportional to the area of the horizon [1, 2], i.e., $T = \frac{\kappa}{2\pi}$, $S = \frac{A}{4G}$. Also this temperature entropy and the mass of the BH are related by the first law of thermodynamics [3]. Further, the equivalence of the thermodynamical quantities with the geometry of the horizon leads to speculate some inherent relationship between thermodynamical laws and Einstein equations. This speculation comes into true in 1995 when Jacobson [4] was able to formulate Einstein's equations from the Clausius relation ($\delta Q = TdS$) using local Rindler Causal horizons with T , the Unruh temperature as seen by an accelerated observer just inside the horizon. The equivalence in other way round was shown by Padmanabhan [5] for a general spherical symmetric space time. However in other gravity theories, such type of equivalence is not possible. Eling et al [6] showed that the usual Clausius relation and the entropy assumption $S = \alpha f'(r)A$ (or $S = \alpha F(\phi)A$) for $f(R)$ -gravity (or scalar tensor gravity) do not give the correct equations of motion-on entropy production term has to be added to the Clausius relation and as a result there will be non-equilibrium thermodynamics of space-time [6, 7].

Similarly, in the context of BH thermodynamics most studies of BH thermodynamics are concentrated to stationary BHs. It is speculated that thermodynamics of dynamical (i.e., non stationary) BH is related to the non-equilibrium thermodynamics of the universe. Hayward [8] initiated the study of the thermodynamics of dynamical BH by proposing "Unified first law". He introduced the idea of the trapping horizon and was able to show the equivalence of Einstein equations and unified first law. He formulated the first law of thermodynamics for a dynamical BH by projecting unified first law along the trapping horizon. Also projecting along the tangent to the trapping horizon he was able to formulate the Clausius relation. Subsequently, Cai et al [7] studied in details the thermodynamics of FRW universe starting from the unified first law, in Einstein gravity, Lovelock gravity and in scalar-tensor theory of gravity and showed the necessity of introducing entropy production term in scalar tensor theory.

In this small article we study the unified first law in details and reveal some of its properties in the form of lemmas.

Mathematically, the unified first law can be written as

$$dE = A\psi + WdV \quad (1)$$

Here,

$$E = \frac{R}{2G} (1 - h^{ab} \partial_a R \partial_b R), \quad (2)$$

is known as Misner-sharp energy. this is the total energy inside the sphere of radius R (known as area radius). The energy supply ψ and the work function W are given by [7, 8]

$$\Psi = \psi_a dx^a, \quad W = -\frac{1}{2} \text{Trace}(T) \quad (3)$$

with

$$\psi_a = T_a^b \partial_b R + \partial_a R \quad (4)$$

known as energy-supply vector. The four dimensional metric is written in the form

$$ds^2 = h_{ab} dx^a dx^b + R^2 d\Omega^2 \quad (5)$$

where R is the radius of the 2-sphere, h_{ab} is the metric on the 2-D hyper surface normal to the spherical surface of symmetry. A and V are the area and the volume of the 2-sphere of radius R . In the above equations (3) and (4) T_a^b is the projection of the energy momentum tensor normal to the spherical surface and trace is taken over normal 2D hyper surface.

Now projecting the unified first law of thermodynamics along the tangent to the trapping horizon gives the first law of BH thermodynamics as [7, 9].

$$\langle dE, z \rangle = \frac{\kappa}{8\pi G} \langle dA, z \rangle + W \langle dV, z \rangle \quad (6)$$

where κ and z are respectively the surface gravity and tangent vector to the trapping horizon. Also the Clausius relation has the form,

$$\langle A\psi, z \rangle = \frac{\kappa}{8\pi G} \langle dA, z \rangle \quad (7)$$

For simplicity, we consider homogeneous and isotropic FRW model of the universe and the line element is written in the form of equation (5). Then we have,

$$h_{ab} = \text{diag} \left(-1, \frac{a^2}{1 - kr^2} \right) \quad (8)$$

and $R = ar$, the area radius.

The explicit form of different thermodynamical parameters are

$$\begin{aligned} E &= \frac{R^3}{2G} \left(H^2 + \frac{k}{a^2} \right) \\ W &= \frac{1}{2} (\rho - p) \\ \psi_0 &= -\frac{1}{2} (\rho + p) H R \\ \psi_1 &= \frac{1}{2} (\rho + p) a \\ \kappa &= \frac{1}{2\sqrt{-h}} \partial_a (\sqrt{-h} h^{ab} \partial_b R) = \frac{2\pi R}{3} (3p - \rho) \end{aligned} \quad (9)$$

where we have assumed perfect fluid as the matter in the universe.

The two null vectors defined along the normal to the spherical surface of symmetry are given by,

$$\xi_+ = \partial_+ = -\sqrt{2} \left(\partial_t - \frac{\sqrt{1-kr^2}}{a} \partial_r \right) \quad , \quad \xi_- = \partial_- = -\sqrt{2} \left(\partial_t + \frac{\sqrt{1-kr^2}}{a} \partial_r \right) \quad (10)$$

and the trapping horizon is defined as

$$h^{ab} \partial_a R = 0 \quad (11)$$

i.e.,

$$R_T = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}} = R_A,$$

the apparent horizon. Now we shall prove the following lemmas :

Lemma I : Trapping horizons (if exists) are always apparent horizons but not the converse.

Proof : The dynamical apparent horizon is the marginally trapped surface with vanishing expansion and is defined as a sphere of radius R_A satisfying

$$h^{ab} \partial_a R \partial_b R = 0 \quad (12)$$

A trapping horizon is a hyper surface foliated by marginal spheres and is mathematically defined as

$$\theta_+ R = \frac{1}{R} \partial_+ R = 0 \quad (13)$$

i.e., in the form of the metric (5) we have

$$h^{ab} \partial_a R = 0 \quad (14)$$

Here θ_+ is the expansion of the null congruence $\xi_+ = \text{constant}$.

Hence

$$h^{ab} \partial_a R \partial_b R = (h^{ab} \partial_a R) \partial_b R = 0$$

But $h^{ab} \partial_a R \partial_b R = 0$ does not imply $h^{ab} \partial_a R = 0$

Thus the trapping horizon is always implies an apparent horizon but not the converse.

Lemma II : Unified first law and both the Friedman equations are equivalent on any spherical surface of symmetry

Proof : From equation (9)

$$dE = \frac{1}{2G} \left[3 \left(H^2 + \frac{k}{a^2} \right) a R^2 (H dt + dr) + 2 H R^3 \left(\dot{H} - \frac{k}{a^2} \right) dt \right] \quad (15)$$

$$\Psi = \psi_0 dt + \psi_1 dr = \frac{1}{2} (\rho + p) [-H R dt + a dr] \quad (16)$$

So

$$A \Psi + W dV = 2\pi R^2 (\rho + p) [-H R dt + a dr] + 2\pi R^2 (\rho - p) [H R dt + a dr]$$

$$= 4\pi R^2 [-pHRdt + a\rho dr]$$

Hence equating coefficients of dt and dr in the unified first law (1) we have (after some simplification)

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \quad (17)$$

and

$$\dot{H} - \frac{k}{a^2} = -4\pi G(\rho + p), \quad (18)$$

the two Friedman equations at any R . Hence unified first law and Friedman equations are equivalent on any spherical surface.

Note : In ref [7] Cai et al stated that (see remarks after eq. (4.14)) Unified first law is just an identity related to the (0, 0)-th component of Einstein equations. But in the above lemma we have shown that both the Friedman equations are derivable from the unified first law and vice versa.

Lemma III : The validity of the Clausius relation depends on the choice of the tangent vector \mathbf{z} .

Proof : According to Hayward the Clausius relation has the form :

$$\langle A\Psi, \mathbf{z} \rangle = \frac{\kappa}{8\pi G} \langle dA, \mathbf{z} \rangle$$

i.e.,

$$2\pi R^2(\rho + p) \left[-HRz_1 + \sqrt{1 - kr^2}z_2 \right] = \frac{\kappa}{G}R \left[RH z_1 + \sqrt{1 - kr^2}z_2 \right]$$

i.e.,

$$\frac{z_1}{z_2} = \frac{\sqrt{1 - \frac{k}{a^2}R^2}}{RH} \left(\frac{2\rho}{\rho + 3p} \right)$$

Thus if the coefficients z_1, z_2 of the tangent vector satisfy the above relation then only Clausius relation holds, i.e., at a definite spherical hyper surface only for a particular tangent vector Clausius relation holds.

Lemma IV : The first law of BH thermodynamics is equivalent to Clausius relation at any spherical surface.

Proof : The first law of BH thermodynamics in Hayward's formalism can be written as

$$\langle dE, \mathbf{z} \rangle = \frac{\kappa}{8\pi G} \langle dA, \mathbf{z} \rangle + W \langle dV, \mathbf{z} \rangle \quad (19)$$

Now writing explicitly both sides we have (we have after some simplifications)

$$\frac{z_1}{z_2} = \frac{\sqrt{1 - \frac{K}{a^2}R^2}}{RH} \cdot \frac{2\rho}{(\rho + 3p)}$$

Thus on a particular spherical surface and for the same tangent vector both first law of BH thermodynamics and Clausius relation hold.

The present study shows that although the unified first law is expressed in terms of thermodynamical parameters, it is essentially an alternative form of Einstein field equations. It is found that projecting this unified first law along tangent to the spherical surface of symmetry gives the equivalence of 1st law of BH thermodynamics and Clausius relation but it holds only for a specific choice of the tangent vector which may be different for different spherical surfaces. For future work, it will be nice to examine the equivalence of unified first law and the Einstein field equations for general space-time model.

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